A DESCRIPTION OF THE QUANTUM SUPERALGEBRA U_q[OSP(2N+1/2M)] VIA GREEN GENERATORS

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1. Introduction. Outline of the results

In the present talk I'll describe the orthosymplectic Lie superalgebra $osp(2n+1/2m)$ and also its $q-$deformed analogue $U_q[osp(2n+1/2m)]$ in terms of a new set of generators, called Green generators. These generators are very different form the well known Chevalley generators. Let me underline from the very beginning that I am not going to consider new deformation of $U_q[osp(2n+1/2m)]$. The deformation will be the known Hopf algebra deformation as given, for instance, in [1-4]. The description, however, will be given in terms of new free generators.

For me personally the interest in the construction stems from the observation that the Green generators are of a direct physical significance. In a certain representation of $osp(2n+1/2m)$ part of these generators are Bose operators, whereas the rest are Fermi operators. Considered as elements from the universal enveloping algebra, the Green generators are para-Bose and para-Fermi operators [5]. To begin with I'll state the final result. It is contained in the following

Theorem. *$U_q[osp(2n+1/2m)]$ is an associative superalgebra with 1, generators $a_i^{\pm}$, L_i, $\bar{L}_i \equiv L_i^{-1}$, $i=1,2,\ldots,m+n=N$, relations ($\xi,\eta=\pm$ or ± 1, $\bar{q}\equiv q^{-1}$)*

$$\begin{aligned}
&L_iL_i^{-1}=L_i^{-1}L_i=1, \quad L_iL_j=L_jL_i,\\
&L_ia_j^{\pm}=q^{\pm\delta_{ij}(-1)^{\langle i\rangle}}a_j^{\pm},\\
&[\![a_i^-,a_i^+]\!]=-2\frac{L_i-\bar{L}_i}{q-\bar{q}},\\
&[\![[\![a_i^{\eta},a_{i+\xi}^{-\eta}]\!],a_j^{\eta}]\!]_{q^{-\xi(-1)^{\langle i\rangle}\delta_{ij}}}=2(\eta)^{\langle j\rangle}\delta_{j,i+\xi}L_j^{-\xi\eta}a_i^{\eta},\\
&[[a_{N-1}^{\xi},a_N^{\xi}],a_N^{\xi}]_{\bar{q}}=0.
\end{aligned} \tag{2}$$

and $\mathbf{Z}_2$-grading induced from

$$\deg(L_i)=\bar{0},\quad \deg(a_i^{\pm})=\langle i\rangle\equiv\begin{cases}\bar{1}, & \textit{for } i\le m\\ \bar{0}, & \textit{for } i>m.\end{cases} \tag{3}$$

Here and throughout

$$[x,y]_q=xy-qyx,\quad \{x,y\}_q=xy+qyx,\quad [\![x,y]\!]_q=xy-(-1)^{\deg(x)\deg(y)}qyx. \tag{4}$$

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This theorem extends the results of several previous publications. The first deformation of one pair of para-Bose operators was given independently in [6] and [7]. The second paper includes also all Hopf algebra operations. This result was generalized to any number of parabosons in [8, 9], including some representations in the root of unity case [10]. A similar problem for any number of parafermions was solved in [11]. The deformation of one pair of parafermions and one pair of parabosons was carried out in [12]. Finally, the nondeformed version of the present investigation is given in [13].

The plan of the exposition will be the following. First in Sect 2 I'll recall the definition of the orthosimplectic Lie superalgebra (LS) $osp(2n+1/2m)$ in a matrix form. As next steps, a description of its universal enveloping algebra in terms of operators, called preoscillator generators (Sect. 3), and via Green generators (Sect. 4) will be given. Finally, in Sect. 5 the deformed algbra will be considered and some indications of how the proof of the Theorem goes will be mentioned.

2. Definition of $osp(2n+1/2m)$ in a matrix form [14]

The Lie superalgebra $osp(2n+1/2m)$ can be defined as the set of all $(2n+2m+1)\times(2n+2m+1)$ matrices of the form (T=transposition)

$$\begin{pmatrix} a & b & u & x & x_1 \\ c & -a^T & v & y & y_1 \\ -v^T & -u^T & 0 & z & z_1 \\ y_1^T & x_1^T & z_1^T & d & e \\ -y^T & -x^T & -z^T & f & -d^T \end{pmatrix}, \tag{5}$$

where a is any $n\times n$ matrix, b and c are skew symmetric $n\times n$ matrices, d is any $m\times m$ matrix, e and f are symmetric $m\times m$ matrices, x, x_1, y, y_1 are $n\times m$ matrices, u and v are $n\times 1$ columns, z, z_1 are $1\times m$ rows. The even subalgebra consists of all matrices with $x=x_1=y=y_1=z=z_1=0$, namely

$$\begin{pmatrix} a & b & u & 0 & 0 \\ c & -a^T & v & 0 & 0 \\ -v^T & -u^T & 0 & 0 & 0 \\ 0 & 0 & 0 & d & e \\ 0 & 0 & 0 & f & -d^T \end{pmatrix}, \tag{6}$$

and it is isomorphic to the Lie algebra $so(2n+1)\oplus sp(2m)$. The odd subspace is given with all matrices

$$\begin{pmatrix} 0 & 0 & 0 & x & x_1 \\ 0 & 0 & 0 & y & y_1 \\ 0 & 0 & 0 & z & z_1 \\ y_1^T & x_1^T & z_1^T & 0 & 0 \\ -y^T & -x^T & -z^T & 0 & 0 \end{pmatrix}. \tag{7}$$

The product (= the supercommutator) is defined on any two homogeneous elements a and b as

$$[\![a,b]\!] = ab - (-1)^{deg(a)deg(b)}ba. \tag{8}$$

Let $L(n/m)$ be the $2(n+m)$-dimensional $\mathbf{Z}_2$-graded subspace, consisting of all matrices

$$\begin{pmatrix} 0 & 0 & u & 0 & 0 \\ 0 & 0 & v & 0 & 0 \\ -v^T & -u^T & 0 & z & z_1 \\ 0 & 0 & z_1^T & 0 & 0 \\ 0 & 0 & -z^T & 0 & 0 \end{pmatrix}. \tag{9}$$

Label the rows and the columns with the indices $A, B = -2n, -2n+1, \ldots, -2, -1, 0, 1, 2, \ldots, 2m$ and let e_{AB} be a matrix with 1 at the intersection of the A^{th} row and the B^{th} column and zero elsewhere. Then the following elements (matrices) constitute a basis in $L(n/m)$:

$$\begin{aligned} &a_i^- \equiv B_i^- = \sqrt{2}(e_{0,i} - e_{i+m,0}), \quad a_i^+ \equiv B_i^+ = \sqrt{2}(e_{0,i+m} + e_{i,0}), \quad i = 1, \ldots, m, \\ &a_{j+m}^- \equiv F_j^- = \sqrt{2}(e_{-j,0} - e_{0,-j-n}), \quad a_{j+m}^+ \equiv F_j^+ = \sqrt{2}(e_{0,-j} - e_{-j-n,0}), \quad j = 1, \ldots, n, \end{aligned} \tag{10}$$

with $\deg(a_i^\pm) = \langle i \rangle$.

Proposition 1. *The LS* $osp(2n+1/2m)$ *is generated from* $a_i^\pm,\ i = 1, \ldots, m+n \equiv N$.

It is straightforward to show that

$$osp(2n+1/2m) = lin.env.\{a_i^\xi,\ [\![a_j^\eta, a_k^\varepsilon]\!] | i,j,k = 1, \ldots, N, \quad \xi, \eta, \varepsilon = \pm\}. \tag{11}$$

Hence any further supercommutator between $a_i^\xi,\ [\![a_j^\eta, a_k^\varepsilon]\!],\ \ \xi, \eta, \varepsilon = \pm$, is a linear combination of the same type elements. A more precise computation gives:

$$[\![[\![a_i^\xi, a_j^\eta]\!], a_k^\epsilon]\!] = 2\epsilon^{\langle k \rangle}\delta_{jk}\delta_{\epsilon,-\eta}a_i^\xi - 2\epsilon^{\langle k \rangle}(-1)^{\langle j \rangle\langle k \rangle}\delta_{ik}\delta_{\epsilon,-\xi}a_j^\eta. \tag{12}$$

Eqs. (12) are among the supercommutation relations of all Cartan-Weyl generators

$$a_i^\xi,\ [\![a_j^\eta, a_k^\varepsilon]\!], \quad i,j,k = 1, \ldots, N, \quad \xi, \eta, \varepsilon = \pm. \tag{13}$$

The rest of the supercommutation relations follow from (12) and the (graded) Jacoby identity ($i,j,k,l = 1, \ldots, N, \quad \xi, \eta, \varepsilon, \varphi = \pm$):

$$\begin{aligned} [\![[\![a_i^\xi, a_j^\eta]\!], [\![a_k^\epsilon, a_l^\varphi]\!]]\!] = {}& 2\epsilon^{\langle k \rangle}\delta_{jk}\delta_{\epsilon,-\eta}[\![a_i^\xi, a_l^\varphi]\!] - 2\epsilon^{\langle k \rangle}(-1)^{\langle j \rangle\langle k \rangle}\delta_{ik}\delta_{\epsilon,-\xi}[\![a_j^\eta, a_l^\varphi]\!] \\ & - 2\varphi^{\langle l \rangle}(-1)^{\langle j \rangle\langle k \rangle}\delta_{jl}\delta_{\varphi,-\eta}[\![a_i^\xi, a_k^\epsilon]\!] + 2\varphi^{\langle l \rangle}(-1)^{\langle i \rangle\langle j \rangle + \langle i \rangle\langle k \rangle}\delta_{il}\delta_{\varphi,-\xi}[\![a_j^\eta, a_k^\epsilon]\!]. \end{aligned} \tag{14}$$

3. Description of $U[osp(2n+1/2m)]$ via preoscillator generators

The relations (12) are representation independent. More precisely, the universal enveloping algebra (UEA) $U[osp(2n+1/2m)]$ of $osp(2n+1/2m)$ is by definition the (free) associative algebra with 1 of the indeterminates $a_1^\pm, a_2^\pm, \ldots, a_{m+n}^\pm \equiv a_N^\pm$, subject to the relations (12) and (14). Since however Eqs. (14) follow from (12), we have

Proposition 2. *(1)* $U[osp(2n+1/2m)]$ *is the associative unital algebra with generators*

$$a_1^\pm, a_2^\pm, \ldots, a_{m-1}^\pm, a_m^\pm, a_{m+1}^\pm, \ldots, a_{m+n}^\pm \equiv a_N^\pm, \tag{15}$$

relations

$$[\![[\![a_i^\xi, a_j^\eta]\!], a_k^\epsilon]\!] = 2\epsilon^{\langle k \rangle}\delta_{jk}\delta_{\epsilon,-\eta}a_i^\xi - 2\epsilon^{\langle k \rangle}(-1)^{\langle j \rangle\langle k \rangle}\delta_{ik}\delta_{\epsilon,-\xi}a_j^\eta \tag{16}$$

and $\mathbf{Z}_2$*-grading induced from*

$$\deg(a_i^\pm) = \langle i \rangle. \tag{17}$$

(2)

$$osp(2n+1/2m) = lin.env.\{a_i^\xi,\ [\![a_j^\eta, a_k^\varepsilon]\!] | i,j,k = 1, \ldots, N, \quad \xi, \eta, \varepsilon = \pm\}, \tag{18}$$

with a natural supercommutator (turning every associative superalgebra into a Lie superalgebra):

$$[\![a, b]\!] = ab - (-1)^{deg(a)deg(b)}ba. \tag{19}$$

The above proposition gives a definition of $U[osp(2n+1/2m)]$ in terms of a new set of generators, which are very different from the Chevalley generators. The relevance of the generators $a_i^\pm$ stems from the following observation. The operators $a_i^\pm \equiv B_i^\pm$ with $i=1,\ldots,m$ satisfy the triple relations

$$[\{B_i^\xi, B_j^\eta\}, B_k^\varepsilon] = 2\varepsilon\delta_{jk}\delta_{\varepsilon,-\eta}B_i^\xi + 2\varepsilon\delta_{ik}\delta_{\varepsilon,-\xi}B_j^\eta, \tag{20}$$

whereas $a_{i+m}^\pm \equiv F_i^\pm$ with $i=1,\ldots,n$ yields:

$$[[F_i^\xi, F_j^\eta], F_k^\varepsilon] = 2\delta_{jk}\delta_{\varepsilon,-\eta}F_i^\xi - 2\delta_{ik}\delta_{\varepsilon,-\xi}F_j^\eta. \tag{21}$$

The relations (20) and (21) are known in quantum field theory. They are defining relations for para-Bose and for para-Fermi creation and annihilation operators, respectively [5]. The para-Fermi operators generate the Lie algebra $so(2n+1)$ [15], whereas m pairs of para-Bose operators generate a Lie superalgebra [16], which is isomorphic to $osp(1/2m)$ [17].

In the Fock representation the para-Bose (resp. the para-Fermi) operators become usual Bose (resp. Fermi) operators, namely oscillator generators. For this reason we call the operators (15) *preoscillator (creation and annihilation) generators* of $U[osp(2n+1/2m)]$ (resp. of $osp(2n+1/2m)$). The preoscillator generators give an alternative to the Chevalley description of $U[osp(2n+1/2m)]$.

Observe that in this setting the para-Bose (resp. the Bose) operators are odd, whereas the para-Fermi (and the Fermi) operators are even generators.

Coming back to the defining relations (16) of the preoscillator generators we note that they define a linear map

$$L(n/m)\otimes L(n/m)\otimes L(n/m) \to L(n/m), \tag{22}$$

which identifies $osp(2n+1/2m)$ also as a Lie-supertriple system, an approach which was recently developed in [18].

Our purpose is to quantize $U[osp(2n+1/2m)]$ via the preoscillator creation and annihilation operators. This is however difficult to be done directly via the relations (16). Therefore in the next section we select a subset of relations from (16), which describe completely $U[osp(2n+1/2m)]$, and which are convenient for quantization.

4. Description of $U[osp(2n+1/2m)]$ via Green generators [13]

Proposition 3. *$U[osp(2n+1/2m)]$ is an associative unital superalgebra with generators*

$$a_1^\pm, a_2^\pm, \ldots, a_{m-1}^\pm, a_m^\pm, a_{m+1}^\pm, \ldots, a_{m+n}^\pm \equiv a_N^\pm, \tag{23}$$

referred as to Green generators, relations ($\xi,\ \eta = \pm$ *or* ± 1)

$$\begin{aligned} &[\![[\![a_i^\eta, a_j^{-\eta}]\!], a_k^\eta]\!] = 2\eta^{\langle k\rangle}\delta_{jk}a_i^\eta, \quad |i-j|\le 1, \quad \eta=\pm, \\ &[[a_{N-1}^\eta, a_N^\eta], a_N^\eta] = 0, \quad \eta = \pm, \end{aligned} \tag{24}$$

and $\mathbf{Z_2}$*-grading*

$$\deg(a_i^\pm) = \langle i\rangle. \tag{25}$$

The Green generators (23) are the preoscillator generators of $U[osp(2n+1/2m)]$.

In order to indicate how the proof can be done we recall the Chevalley definition of $U[osp(2n+1/2m)]$ and write down explicit relations between the Green and the Chevalley generators. Let $(\alpha_{ij}),\ i,j=1,\ldots,N$ be an $N\times N$ symmetric Cartan matrix chosen as:

$$(a_{ij}) = (-1)^{\langle j\rangle}\delta_{i+1,j} + (-1)^{\langle i\rangle}\delta_{i,j+1} - [(-1)^{\langle j+1\rangle} + (-1)^{\langle j\rangle}]\delta_{ij} + \delta_{i,m+n}\delta_{j,m+n}. \tag{26}$$

For instance the Cartan matrix of $B(4/4) \equiv osp(9/8)$ is 8×8 dimensional matrix:

$$(a_{ij}) = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}. \tag{27}$$

Then $U[osp(2n+1/2m)]$ is defined as an associative superalgebra with 1 in terms of a number of generators subject to a number of relations. The generators are the Chevalley generators $h_i,\ e_i,\ f_i,\ i = 1,\ldots,N$; the relations are the Cartan-Kac relations

$$[h_i, h_j] = 0,\ [h_i, e_j] = a_{ij}e_j,\ [h_i, f_j] = -a_{ij}f_j,\ [\![e_i, f_j]\!] = \delta_{ij}h_i, \tag{28}$$

the $e-$Serre relations

$$\begin{aligned} &[e_i, e_j] = 0,\ \text{for } |i-j| > 1;\ \ [\![e_i, [e_i, e_{i\pm1}]]\!] = 0,\ i \neq N; \\ &\{[e_{m-1}, e_m], [e_m, e_{m+1}]\} = 0;\ [e_N, [e_N, [e_N, e_{N-1}]]] = 0; \end{aligned} \tag{29a}$$

and the $f-$Serre relations

$$\begin{aligned} &[f_i, f_j] = 0,\ \text{for } |i-j| > 1;\ \ [\![f_i, [f_i, f_{i\pm1}]]\!] = 0,\ i \neq N; \\ &\{[f_{m-1}, f_m], [f_m, f_{m+1}]\} = 0;\ [f_N, [f_N, [f_N, f_{N-1}]]] = 0. \end{aligned} \tag{29b}$$

The grading on $U[osp(2n+1/2m)]$ is induced from: $\deg(e_m) = \deg(f_m) = \bar{1},\ \deg(e_i) = \deg(f_i) = \bar{0}$ for $i \neq m$.

The expressions of the Green generators in terms of the Chevalley generators read $(i = 1,\ldots,N-1)$:

$$\begin{aligned} &a_i^- = (-1)^{(m-i)\langle i\rangle}\sqrt{2}[e_i, [e_{i+1}, [\ldots, [e_{N-2}, [e_{N-1}, e_N]]\ldots]]], \quad a_N^- = \sqrt{2}e_N, \\ &a_i^+ = -\sqrt{2}[f_i, [f_{i+1}, [\ldots, [f_{N-2}, [f_{N-1}, f_N]]\ldots]]], \quad a_N^+ = -\sqrt{2}f_N. \end{aligned} \tag{30}$$

Then one proves that $a_i^{\pm}$ generate $U[osp(2n+1/2m)]$ $(i = 1,\ldots,N-1)$,

$$\begin{aligned} &h_i = \frac{1}{2}[\![a_{i+1}^-, a_{i+1}^+]\!] - \frac{1}{2}[\![a_i^-, a_i^+]\!], \quad h_N = -\frac{1}{2}[\![a_N^-, a_N^+]\!], \\ &e_i = \frac{1}{2}[\![a_i^-, a_{i+1}^+]\!], \quad e_N = \frac{1}{\sqrt{2}}a_N^-, \\ &f_i = \frac{1}{2}[\![a_i^+, a_{i+1}^-]\!], \quad f_N = -\frac{1}{\sqrt{2}}a_N^+, \end{aligned} \tag{31}$$

and that the Cartan-Kac and the Serre relations follow from (24) and (31).

5. Description of $U_q[osp(2n+1/2m)]$ via deformed Green generators

The q-deformed superalgebra $U_q[osp(2n+1/2m)]$, a Hopf superalgebra, is by now a classical concept. See, for instance, [1-4] where all Hopf algebra operations are explicetly given. Here, following [4], we write only the algebra operations.

Proposition 4. *$U_q[osp(2n+1/2m)]$ is an associative unital algebra with Chevalley generators $e_i,\ f_i,\ k_i = q^{h_i},\ \bar{k}_i \equiv k_i^{-1} = q^{-h_i},\ i = 1, \ldots, N$, which satisfy the Cartan-Kac relations*

$$\begin{aligned}
&k_i k_i^{-1} = k_i^{-1} k_i = 1, \quad k_i k_j = k_j k_i, \\
&k_i e_j = q^{\alpha_{ij}} e_j k_i, \quad k_i f_j = q^{-\alpha_{ij}} f_j k_i, \\
&[\![e_i, f_j]\!] = \delta_{ij} \frac{k_i - \bar{k}_i}{q - \bar{q}},
\end{aligned} \tag{32}$$

the $e-$Serre relations

$$\begin{aligned}
&(e1) \quad [\![e_i, e_j]\!] = 0, \quad |i-j| \neq 1, \\
&(e2) \quad [e_i, [e_i, e_{i\pm1}]_{\bar{q}}]_q \equiv [e_i, [e_i, e_{i\pm1}]_q]_{\bar{q}} = 0, \quad i \neq m, \ \ i \neq N \\
&(e3) \quad \{[e_m, e_{m-1}]_q, [e_m, e_{m+1}]_{\bar{q}}\} = 0, \\
&(e4) \quad [e_N, [e_N, [e_N, e_{N-1}]_{\bar{q}}]]_q \equiv [e_N, [e_N, [e_N, e_{N-1}]_q]]_{\bar{q}} = 0,
\end{aligned} \tag{33}$$

and the $f-$Serre relations

$$\begin{aligned}
&(f1) \quad [\![f_i, f_j]\!] = 0, \quad |i-j| \neq 1, \\
&(f2) \quad [f_i, [f_i, f_{i\pm1}]_{\bar{q}}]_q \equiv [f_i, [f_i, f_{i\pm1}]_q]_{\bar{q}} = 0, \quad i \neq m, \ \ i \neq N \\
&(f3) \quad \{[f_m, f_{m-1}]_q, [f_m, f_{m+1}]_{\bar{q}}\} = 0, \\
&(f4) \quad [f_N, [f_N, [f_N, f_{N-1}]_{\bar{q}}]]_q \equiv [f_N, [f_N, [f_N, f_{N-1}]_q]]_{\bar{q}} = 0.
\end{aligned} \tag{34}$$

The $(e3)$ and $(f3)$ Serre relations are the additional Serre relations [19-21], which were initially omitted.

We are now ready to state our main result, given also in the Introduction.

Theorem. *$U_q[osp(2n+1/2m)]$ is an associative superalgebra with 1, generators $a_i^{\pm},\ L_i,\ \bar{L}_i \equiv L_i^{-1}$, $i = 1, 2, \ldots, m+n = N$, relations ($\xi, \eta = \pm$ or ± 1, $\bar{q} \equiv q^{-1}$)*

$$\begin{aligned}
&L_i L_i^{-1} = L_i^{-1} L_i = 1, \quad L_i L_j = L_j L_i, \\
&L_i a_j^{\pm} = q^{\pm \delta_{ij} (-1)^{\langle i \rangle}} a_j^{\pm}, \\
&[\![a_i^-, a_i^+]\!] = -2 \frac{L_i - \bar{L}_i}{q - \bar{q}}, \\
&[\![[\![a_i^\eta, a_{i+\xi}^{-\eta}]\!], a_j^\eta]\!]_{q^{-\xi(-1)^{\langle i \rangle} \delta_{ij}}} = 2(\eta)^{\langle j \rangle} \delta_{j,i+\xi} L_j^{-\xi\eta} a_i^\eta, \\
&[[a_{N-1}^\xi, a_N^\xi], a_N^\xi]_{\bar{q}} = 0.
\end{aligned} \tag{35}$$

and $\mathbf{Z}_2$-grading $\deg(L_i) = \bar{0},\ \deg(a_i^{\pm}) = \langle i \rangle$.

The expressions of $a_i^{\pm}$ and L_i via the Chevalley generators read ($i = 1, \ldots, N-1$):

$$\begin{aligned}
&L_i = k_i k_{i+1} \ldots k_N \ \text{(including } i = N\text{)}, \\
&a_i^- = (-1)^{(m-i)\langle i \rangle} \sqrt{2} [e_i, [e_{i+1}, [\ldots, [e_{N-2}, [e_{N-1}, e_N]_{q_{N-1}}]_{q_{N-2}} \ldots]_{q_{i+2}}]_{q_{i+1}}]_{q_i}, \quad a_N^- = \sqrt{2} e_N, \\
&a_i^+ = (-1)^{N-i+1} \sqrt{2} [[[\ldots [f_N, f_{N-1}]_{\bar{q}_{N-1}}, f_{N-2}]_{\bar{q}_{N-2}} \ldots]_{\bar{q}_{i+2}}, f_{i+1}]_{\bar{q}_{i+1}}, f_i]_{\bar{q}_i}, \quad a_N^+ = -\sqrt{2} f_N,
\end{aligned} \tag{36}$$

where

$$q_i = \bar{q},\ i = 1, \ldots, m-1; \quad q_i = q,\ i = m, \ldots, N.$$

The next result is essential for the proof of the Theorem.

Proposition 5. *The following relations hold:*

$$1. \quad [\![e_i, a_j^+]\!] = -\delta_{ij}(-1)^{\langle i+1\rangle} k_i a_{i+1}^+, \quad i \neq N, \tag{37}$$

$$2. \quad [\![a_j, f_i]\!] = \delta_{ij} a_{i+1}^- \bar{k}_i, \quad i \neq N, \tag{38}$$

$$3. \quad [\![e_i, a_j^-]\!] = 0, \ \ if\ i < j-1 \ or\ i > j, \quad i \neq N, \tag{39a}$$

$$[\![e_i, a_{i+1}^-]\!]_{q_i} = (-1)^{\langle i+1\rangle} a_i^-, \quad i \neq N, \tag{39b}$$

$$[\![e_i, a_i^-]\!]_{\bar{q}_{i-1}} = 0, \quad i \neq N, \tag{39c}$$

$$4. \quad [\![a_j^+, f_i]\!] = 0, \ \ if\ i < j-1 \ or\ i > j, \quad i \neq N, \tag{40a}$$

$$[\![a_{i+1}^+, f_i]\!]_{\bar{q}_i} = -a_i^+, \quad i \neq N. \tag{40b}$$

$$[\![a_i^+, f_i]\!]_{q_{i-1}} = 0, \quad i \neq N. \tag{40c}$$

Also here one proves that $a_i^\pm$ and $L_i^{\pm 1}$ generate $U_q[osp(2n+1/2m)]$. More precisely ($i = 1, \ldots, N-1$),

$$\begin{aligned}
&k_i = L_i \bar{L}_{i+1}, \quad L_N = k_N,\\
&e_i = \frac{1}{2}\bar{L}_{i+1}[\![a_i^-, a_{i+1}^+]\!], \quad e_N = \frac{1}{\sqrt{2}} a_N^-,\\
&f_i = \frac{1}{2}[\![a_i^+, a_{i+1}^-]\!] L_{i+1}, \quad f_N = -\frac{1}{\sqrt{2}} a_N^+.
\end{aligned} \tag{41}$$

It is a long computation to show, using only the relations (35), that the operators (41) satisfy the Cartan-Kac and the Serre relations. The proof is based on repeated use of nontrivial identities. Here is one of them.

Proposition 6. *If B or C is an even element, then for any values of the parameters x, y, z, t, r, s subject to the relations*

$$x = zs, \ \ y = zr, \ \ t = zsr, \tag{42}$$

the following identity holds:

$$[\![A, [B, C]_x]\!]_y = [\![[\![A, B]\!]_z, C]\!]_t + (-1)^{deg(A)deg(B)} z [\![B, [\![A, C]\!]_r]\!]_s. \tag{43}$$

In particular it is nontrivial to prove that $e_m^2 = 0$, which is one of the Serre relations, or to show that the additional Serre relations ($e3$) and ($f3$) hold.

5. Concluding remarks

The root system of the orthosymplectic Lie superalgebra $osp(2n+1/2m)$ reads:

$$\Delta = \{\xi\varepsilon_i + \eta\varepsilon_j; \xi\varepsilon_i; 2\xi\varepsilon_k\}, \quad i \neq j = 1, \ldots, m+n \equiv N; \ k = 1, \ldots, m; \ \xi, \eta = \pm\}. \tag{44}$$

The roots $\varepsilon_1, \ldots, \varepsilon_N$ are orthogonal with respect to the Killing form on $osp(2n+1/2m)$. The Green generators are the root vectors, corresponding to the orthogonal roots. More precisely, the correspondence reads:

$$a_i^\pm \leftrightarrow \mp\varepsilon_i, \quad i = 1, \ldots, N. \tag{45}$$

Therefore what we have done here is

(1) to describe $U[osp(2n+1/2m)]$ in terms of a "minimal" set of relations among the positive and the negative root vectors, corresponding to the orthogonal roots.

(2) to describe $U_q[osp(2n+1/2m)]$ entirely in terms of deformed "orthogonal" root vectors, namely deformed Green generators.

This is a good opportunity to mention that the canonical quantum statistics and its generalization, the parastatistics, is based on the representation theory of orthosymplectic Lie superalgebras. For instance the Bose operators $B_i^{\pm},\ i=1,\dots,n$ are generators of $osp(1/2n)$ in a particular representation. Similar statement holds for n pairs of Fermi creation and annihilation operators: they are generators of the Lie algebra $so(2n+1)$ in a particular, the Fock representation. Both $osp(1/2n)$ and $so(2n+1)$ are among the superalgebras from the class B in the classification of Kac of the basic Lie superalgebras [14]. Therefore the canonical quantum statistics and its generalization, the parastatistics, could be called *B-statistics*.

One can associate a concept of creation and annihilation operators with every simple Lie algebra [22-24] and presumably also with every basic Lie superalgebra. The creation and the annihilation operators of the Lie superalgebra sl(1/n) were given in [24]. Therefore, parallel to the *B-statistics*, i.e., the parastatistics, there exists *A-statistics*, *C-statistics* and *D-statistics*. The corresponding deformations, certainly, also exist. In fact the *A-statistics* belongs to the class of the exclusion statistics, recently introduced by Haldane [25] in solid state physics.

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